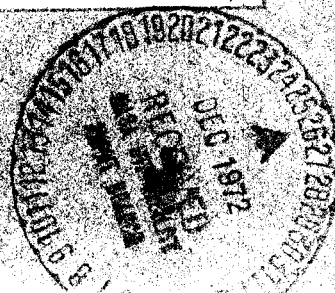


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ORDER, TOPOLOGY AND PREFERENCE

Murat R. Sertel

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ORDER, TOPOLOGY AND PREFERENCE

Murat R. Sertel

This is a reasonably self-contained paper bringing some standard order-related and topological notions, facts and methods to bear on a number of central topics in the theory of preference and, as a natural but henceforth unmentioned correlate, the theory of optimization. Much of the material brought to bear is well-known to economic theorists and even more so to mathematicians. Part of it, however, falls into that growing class of mathematical results motivated by social analysis, and is, I think, new. Such results will be found interspersed throughout Section 3 as extensions of the basics of Nachbin's [9] work. Among such results, however, the two theorems under 3.4 and 3.5 are probably the more important to note.

The topic of Section 4, the theory of preference, is largely but not solely the motivation for the study as a whole. Thus, some of the facts and notions presented before that section are not used at all in Section 4. Sections 1-2 are preparatory. Section 3 exploits some consequences of connectivity, especially from the viewpoint of normally preordered spaces, thus extending the early work of Eilenberg [5] and Nachbin [9] on order and topology.

Section 4 begins with the subsection 4.1 in which Debreu's celebrated first representation theorem [1, Theorem 1, p. 260] is made obsolete by the more general corollary (4.1.1) to theorems (3.4) and (3.5), as a result of which Debreu's assumption of

separability for the space of prospects can be dropped. [Thus, from the viewpoint of economic theory, 3.4 and 3.5 may be looked upon as lemmas aimed at 4.1.1, which is then "elevated" to theorem status.]

The next subsection, 4.2, uses this and [10] to extend the foundations of Gorman's [6] insightful characterization of the structure of preferences. It is indicated how this significantly extends that characterization as a whole by allowing two of Gorman's postulates (separability and arcwise connectivity) for the space of prospects to be relaxed to a much weaker postulate (connectivity).

The content of the brief final subsection, 4.3, is described quite well by its title.

To the reader minimally knowledgeable in topology, this study is mathematically self-contained. In any case, Dugundji's Topology [4] will be our standard reference in this domain. [N.B.: Dugundji uses 'path' for 'arc'.]

Standing Terminology and Notation: The set of real numbers will be denoted by R , while E^k will denote k -dimensional Euclidean space ($k = 0, 1, \dots$). Thus, E^1 will stand for R with the usual (equivalently, the order-) topology. Given a set X , a relation on X will mean a subset $\Gamma \subset X \times X$ of the Cartesian product of X upon itself. A preference relation is a complete transitive relation, and a utility function is simply a real-valued function preserving a preference relation. [See further terminology given in Section 1.]

A set X on which a preference relation is postulated is a space of prospects. The last three underlined terms will seldom be used from here on. 'We' will mean 'you (the reader) and I (the author)'; 'iff' will mean 'if and only if'; 'nbd' will mean 'neighborhood'.

1. Preliminaries

Let $\Gamma \subset X \times X = X^2$ be a relation on a set X . Denote

$$\left. \begin{aligned} \Gamma x &= \{y \in X \mid (x, y) \in \Gamma\} \\ x\Gamma &= \{y \in X \mid (y, x) \in \Gamma\} \end{aligned} \right\} \quad (x \in X).$$

A subset $A \subset X$ is said to be increasing iff $\Gamma A \subset A$, and it is said to be decreasing iff $A\Gamma \subset A$. Clearly, $A \subset X$ is increasing iff its complement, denoted by A^c , is decreasing. Also, it is plain that any intersection and any union of increasing (decreasing) sets is increasing (decreasing). Thus, each set $A \subset X$ determines a unique smallest increasing (decreasing) set, denoted by ΛA ($A\Lambda$), which contains A . In fact, if Γ is transitive, then the relation $\Lambda \subset X^2$ arising from this notation by setting $\Lambda = \Gamma \cup \Delta$ (where $\Delta = \{(x, x) \mid x \in X\}$ is the diagonal of X^2) is the smallest reflexive relation containing Γ , and we have

$\Lambda A = \Gamma A \cup A$, $A\Lambda = A\Gamma \cup A$ ($A \subset X$). [Let $\Omega \subset X^2$. Ω is said to be reflexive iff $\Omega \supset \Delta$, transitive iff $\Omega A \supset \Omega\Omega A$ for all $A \subset X$, antisymmetric iff $\Omega \cap \Omega^{-1} \subset \Delta$, and complete (or total or decisive) iff $\Omega \cup \Omega^{-1} = X^2$, where $\Omega^{-1} = \{(y, x) \mid (x, y) \in \Omega\}$ denotes the converse of Ω . Ω is called a preorder on X iff it is reflexive and transitive; it is called a partial order on X iff it is a preorder on X and antisymmetric; finally, it is called a total order on X iff it is a complete partial order on X].

The proofs of the following two useful facts are entirely straightforward and, hence, omitted.

- 1.1 Exercise: Let $\Gamma \subset X^2$ be a relation on a set X . Then Γ is transitive only if $\Lambda = \Gamma \cup \Delta$ is transitive and iff Γ^{-1} is transitive.
- 1.2 Exercise: Let $\Gamma \subset X^2$ be a relation on a set X , such that Γ^c is complete and antisymmetric. Then
- 1.2.1 Γ is irreflexive (i.e., $\Gamma \cap \Delta = \emptyset$) and so is Γ^{-1} ; in particular, $\Gamma = (\Gamma^c)^{-1} \setminus \Delta$ and $\Gamma^{-1} = \Gamma^c \setminus \Delta$.
- 1.2.2 The three relations Γ , Δ and Γ^{-1} are pairwise disjoint and exhaust X^2 : $X^2 = \Gamma \cup \Delta \cup \Gamma^{-1}$.
- 1.2.3 $\Lambda = (\Gamma^c)^{-1}$.

The following is really a continuation of the above exercises, but is recorded separately because a proof is included.

- 1.3 Proposition: Let $\Gamma \subset X^2$ be a relation on a set X , such that Γ^c is complete and antisymmetric. Then
- 1.3.1 Γ is transitive iff Γ^c is transitive.
- 1.3.2 Γ is transitive iff $\Lambda = \Gamma \cup \Delta$ is transitive.

Proof: (ad 1.3.1 "if"): Assume Γ^c transitive. Then $(\Gamma^c)^{-1}$, too, is transitive; furthermore, by 1.2.1, $\Gamma = (\Gamma^c)^{-1} \setminus \Delta$. Suppose $(x, y), (y, z) \in \Gamma$. To show that $(x, z) \in \Gamma$, it

suffices, then, to show that $(x, z) \notin \Delta$. Suppose $(x, z) \in \Delta$. Then $(z, y), (y, z) \in \Gamma$; hence, $(z, y), (y, z) \in (\Gamma^c)^{-1}$. Thus, antisymmetry of Γ^c implies that $y = z$, contradicting that $(y, z) \in (\Gamma^c)^{-1} \setminus \Delta$. We conclude that $(x, z) \notin \Delta$, i.e., that $(x, z) \in \Gamma$ and that Γ is transitive.

(ad 1.3.1 "only if"): Assume Γ transitive and suppose $(x, y), (y, z) \in \Gamma^c$. Then $(y, x), (z, y) \in (\Gamma^c)^{-1}$. If $y = x$ or $z = y$, then $(x, z) \in \Gamma^c$, leaving nothing to prove. So assume $x \neq y \neq z$. Then, using 1.2.1, we have $(y, x), (z, y) \in (\Gamma^c)^{-1} \setminus \Delta = \Gamma$. Transitivity of Γ thus yields $(z, x) \in \Gamma \subset (\Gamma^c)^{-1}$, so that $(x, z) \in \Gamma^c$, showing Γ^c to be transitive.

(ad 1.3.2): As "only if" is already given in 1.1 (and stated here merely for completeness), we prove "if" only. For that, simply observe that antisymmetry combined with reflexivity for Γ^c yields $\Gamma \cup \Delta = (\Gamma^c)^{-1}$, so that 1.3.1 ensures Γ to be transitive if $(\Gamma^c)^{-1} = \Delta$ is so.

2. Some Basic Order-Topological Facts

The facts and notions presented in this section extend basic propositions demonstrated or notions used by Nachbin [9, pp. 26-27]. A relation $\Gamma \subset X^2$ on a topological space X is said to be semiclosed iff Γx and $x\Gamma$ are both closed for each $x \in X$; it is said to be closed iff it is closed in X^2 .

2.1 Proposition: Let X be a topological space and $\Gamma \subset X^2$ a relation on X .

2.1.1 If for every $(x, y) \in \Gamma^c$ there exist disjoint nbds U and V of x and y , respectively, such that either U is increasing or V is decreasing, then Γ is closed.

2.1.2 If Γ is a closed preorder, then for each $(x, y) \in \Gamma^c$ there exists an increasing nbd U of x and a decreasing nbd V of y such that $U \cap V = \emptyset$.

2.1.3 If Γ is a closed preorder, then it is semiclosed.

Proof: (ad 2.1.1): Suppose $(x, y) \in \Gamma^c$, and that U, V are as described in the hypothesis. Then $U \times V$ is a nbd of (x, y) . Furthermore, if $(u, v) \in (U \times V) \cap \Gamma$, then $u \in v\Gamma$, so that, contrary to assumption, neither can U be increasing, nor V decreasing, since $U \cap V = \emptyset$. Thus, $U \times V \cap \Gamma = \emptyset$, whereby Γ^c is open, i.e., Γ is closed.

(ad 2.1.2): Suppose $(x, y) \in \Gamma^c$, choose a nbd $U' \times V'$ of (x, y) not meeting Γ , and define $U = \Gamma U'$, $V = V'\Gamma$, so that, indeed, U is increasing and V decreasing, while $U \times V \supset U' \times V'$ is a nbd of (x, y) . But $U \cap V = \emptyset$, for if $z \in U$ (so that there exists $u \in U'$ with $z \in \Gamma u$) and $z \in V$ (so that there exists $v \in V'$ with $v \in \Gamma z$), then transitivity of Γ implies that $U' \times V'$ meets Γ (since now $v \in \Gamma z \subset \Gamma \Gamma u \subset \Gamma u$, i.e., $(u, v) \in \Gamma$), a contradiction.

(ad 2.1.2): Given $x \in X$, we show that Γx is closed; imitation shows that $x\Gamma$ is also closed. If $\Gamma x = X$, then

it is closed trivially. So assume $(\Gamma x)^c \neq \emptyset$ and let $y \in (\Gamma x)^c$. Then, by 2.1.2, there exists an increasing nbd U of x and a decreasing nbd V of y with $U \cap V = \emptyset$.

But Γx is the smallest increasing set containing x , so that $\Gamma x \subset U$ and, hence, $\Gamma x \cap V = \emptyset$. This shows that Γx is closed, and completes the proof.

From the fact that $\Lambda = \Gamma \cup \Delta$ and is reflexive, the semiclosedness (closedness) of Δ when X is Frechet (Hausdorff) immediately yields that

2.2 Corollary: Let X be $T_1(T_2)$ and $\Gamma \subset X^2$. If Γ is transitive, then Λ is a preorder, and if Γ is semiclosed (closed), then so is Λ , whereby Λ now becomes a semiclosed (closed) preorder on X .

Furthermore, the conjunction of 2.1.1 and 2.1.2 plainly implies the following

2.3 Corollary: For a preorder $\Gamma \subset X^2$ on a topological space X , being closed is equivalent to the condition that, for each $(x, y) \in \Gamma^c$, x has an increasing nbd disjoint from some decreasing nbd of y .

The next proposition relates our earlier observations to separation properties of X via antisymmetry of $\Gamma \subset X^2$.

2.4 Proposition: Let $\Gamma \subset X^2$ be a partial order on a topological space X . Then X is T_1 (i.e., a Frechet space) if Γ is semiclosed, and X is T_2 (i.e., a Hausdorff space) if Γ is closed.

Proof: Suppose $x, y \in X$ are distinct, so that either

(i) $x \notin \Gamma y$ or (ii) $y \notin \Gamma x$. Assume that Γ is semiclosed.

Suppose that (i) holds. Then $(\Gamma y)^c$ is an open set to which x belongs and y , by the reflexivity of Γ , does not;

similarly, we see that $(x\Gamma)^c$ is open set to which

y belongs and x does not. Also similarly, if (ii) holds,

then $(\Gamma x)^c$ is an open set to which y belongs while x does

not, and $(y\Gamma)^c$ is an open set to which x belongs while y

does not. This shows that X is T_1 , proving the first half

of the proposition. To prove the second half, we note that, if

Γ is closed, then, whether (i) or (ii) holds, 2.1.2 applies,

so that x and y have disjoint nbds, i.e., X is T_2 . This

completes the proof.

3. Consequences of Connectivity

Connectivity plays a great role in the pioneering work of Eilenberg [5] on ordered topological spaces, and we open this section with, essentially, a rewording of one of his early results, including proof for the sake of completeness.

3.1 Proposition: Let X be a connected space and $\Gamma^c \subset X^2$ a semiclosed complete antisymmetric relation. Then Γ is transitive (see also 1.1), hence Γ^c is a total order.

Proof: Suppose $z \in \Gamma y$ and $y \in \Gamma x$ holds for some, $x, y, z \in X$.

Then $\Gamma^c y \subset X \setminus \{z\}$, while 1.2.2 implies that $X \setminus \{z\} = \Gamma z \cup z\Gamma$

which is the union of two disjoint sets Γz and $z\Gamma$, each of

which, by semiclosedness of Γ^c is open. As X is connected, so is $\Gamma^c y$; hence, either $\Gamma^c y \subset \Gamma z$ or $\Gamma^c y \subset z\Gamma$. As $y \in \Gamma^c y$ by reflexivity (from completeness) of Γ^c , and since $y \in z\Gamma$ by assumption, we must have $\Gamma^c y \subset z\Gamma$. But, also by assumption, $x \in y\Gamma = \Gamma^{-1}y$, and 1.2.1 has $\Gamma^{-1}y \subset \Gamma^c y$. Hence, $x \in z\Gamma$, i.e., $z \in \Gamma x$, whereby Γ is seen to be transitive. Then 1.3.1 implies that Γ^c is transitive, hence a total order, as to be shown.

3.2 Proposition: Let $\Gamma \subset X^2$ be a preorder on a connected space X . Then Γ is closed iff it is semiclosed.

Proof: We state "only if" merely for completeness, as it is already given by 2.1.3.

To see "if", assume Γ semiclosed, and suppose $(x, y) \in \Gamma^c$. Combining 2.1.2 with the fact that $\Gamma = \Lambda$ is reflexive and thus Γx is the smallest increasing set containing x while $y\Gamma$ is the smallest decreasing set containing y , we see that $\Gamma x \cap y\Gamma = \emptyset$. Define $Z = (\Gamma x)^c \cap (y\Gamma)^c$. Z cannot be empty, since this would imply $X = Z^c = \Gamma x \cup y\Gamma$, a contradiction of X being connected, since Γx and $y\Gamma$ are closed by assumption and disjoint by demonstration. Let $z \in Z$, and define $U = (z\Gamma)^c$ and $V = (\Gamma z)^c$. By definition of Z , $x \in U$ and $y \in V$, while U and V are open. As the complement of a decreasing (increasing) set, U is increasing (V is decreasing). Thus, 2.1.1 applies, so that Γ is closed. This completes the proof.

3.3 Corollary: Let X be connected and $\Gamma \subset X^2$ a semiclosed complete antisymmetric relation. Then Γ is a closed total order and X is Hausdorff. Furthermore, each point of X is a cutpoint, the infimum or the supremum of X , and, if X is compact, then X is a tree. [N.B. Given X connected, $x \in X$ is called a cutpoint iff $X \setminus \{x\}$ is not connected. Points $y, z \in X$ are said to be separated by (a cutpoint) x iff they belong to distinct components of $X \setminus \{x\}$. A tree is a continuum (i.e., a compact connected T_2 space) whose each two distinct points are separated by some point.]

Proof: That Γ is a closed total order follows directly from the conjunction of 3.1 and 3.2. Then 2.4 implies that X is Hausdorff. Using 1.2.2, $X \setminus \{x\} = \Gamma^c x \cup x \Gamma^c$ for each $x \in X$, where this is a decomposition into two disjoint sets which are open by semiclosedness of Γ . Thus, x is a cutpoint if $\Gamma^c x \neq \emptyset \neq x \Gamma^c$. If $\Gamma^c x = \emptyset$, then $X = \Gamma x$, i.e., x is an infimum and is unique by antisymmetry of Γ . If $x \Gamma^c = \emptyset$, then x is seen, similarly, to be the supremum of X . If X is compact, then it is a continuum, and if x and y are distinct points of X , no generality is lost by assuming $x \in \Gamma^c y$, from which it is straightforward to show that connectedness of X and semiclosedness of Γ implies the existence of a point $z \in X$ with x lying in $\Gamma^c z$ and y

in $z\Gamma^c$, showing that X is a tree and completing the proof.

Actually, the very first consequence in the last corollary implies that if Γ is a semiclosed total order on a connected space X , then Δ is disconnected (since Γ is now closed and Δ is the union, by 1.2.2, of the two disjoint sets Γ^c and $(\Gamma^c)^{-1}$ each of which is open by closedness of Γ). This is a rewording of a "half" of the first of three main theorems of Eilenberg [5, Theorem I, p. 40], the other "half" of which states, conversely, that Δ is disconnected for a connected space X only if X can be endowed a semiclosed total order.

For what follows, we will need Nachbin's [9, p. 28] generalization of the familiar notion of a normal space, namely, that of a normally preordered space. A topological space X equipped with a preorder $\Gamma \subset X^2$ is said to be normally preordered (by Γ) iff, for every two disjoint closed sets $P_0, P_1 \subset X$ such that P_0 is decreasing and P_1 increasing, P_0 and P_1 have disjoint open nbds U_0 and U_1 , respectively, with U_0 decreasing and U_1 increasing. We need some further terminology. Let Γ be a preorder on a topological space X . If $Y \subset X$, then the smallest increasing (decreasing) closed set containing Y will be denoted by $I(Y)$ (respectively, $D(Y)$). If $Y, Z \subset X$, we will write $Y < Z$ to mean that $D(Y) \cap I(Z) = \emptyset$; we will write $Y \ll Z$ to mean that Y and Z have disjoint open nbds U and V , respectively, such that U is decreasing and V increasing. We will use the following simple characterization theorem [9, p.29]:

A topological space X equipped with a preorder is normally preordered iff $Y \ll Z$ whenever $Y < Z$ ($Y, Z \subset X$).

3.4 Theorem: Let X be a connected space completely preordered by a semiclosed relation $\Gamma \subset X^2$. Then X is normally preordered by Γ .

Proof: Let $A, B \subset X$ and suppose $A < B$. Assume $A \neq \emptyset \neq B$, for otherwise $A \ll B$ trivially. Since $D(A)$ and $I(B)$ are closed and disjoint nonempty sets ($A \subset D(A)$ and $B \subset I(B)$, by reflexivity of Γ), connectivity of X clearly implies that there is a point $y \in X$ such that $y \notin D(A) \cup I(B)$.

As Γ is a preorder, the smallest increasing set containing y is Γy and the smallest decreasing set containing y is $y\Gamma$. As Γ is semiclosed, $D(\{y\}) = y\Gamma$ and

$I(\{y\}) = \Gamma y$. By transitivity of Γ , decreasingness of $D(A)$ and increasingness of $I(B)$, $y \notin D(A) \cup I(B)$ implies that

$D(A) \cap \Gamma y = \emptyset = y\Gamma \cap I(B)$, i.e., that $A < \{y\} < B$. Thus, Γ

being semiclosed, $\Gamma^c y$ is an open decreasing nbd of A and

$y\Gamma^c$ and open increasing nbd of B , and $\Gamma^c y \cap y\Gamma^c = \emptyset$ by

transitivity of Γ . Hence, $A \ll B$, showing that X is normally preordered by Γ .

The following theorem of Nachbin [9, Theorem 2, p. 36] gives us Urysohn's famous extension theorem for continuous real-valued functions on a normal space when it is noted that a normal space is simply a space X normally preordered by the

discrete partial order $\Delta \subset X^2$ and X is considered to be normally preordered by Δ .

Theorem: Let X be a space normally preordered by Γ , and let $P \subset X$ be a closed subset such that $f_p: P \rightarrow E^1$ is a bounded real-valued continuous function preserving Γ [i.e., $y \in \Gamma x \Rightarrow f_p(y) \geq f_p(x)$]. For each $\lambda \in E^1$, denote $A(\lambda) = \{x \in P \mid f_p(x) \leq \lambda\}$ and $B(\lambda) = \{x \in P \mid f_p(x) \geq \lambda\}$. Then f_p can be extended to some bounded (real-valued) continuous Γ -preserving function $f: X \rightarrow E^1$ iff $A(\lambda) < B(\lambda')$ whenever $\lambda < \lambda'$.

Our immediate motivation for recording this theorem is its use in proving the following

3.5 Theorem: Let X be a space normally preordered by a relation $\Gamma \subset X^2$ such that $\Sigma = \Gamma \cap \Gamma^{-1}$ is semiclosed. Then there exists a bounded continuous real-valued function $f: X \rightarrow E^1$ preserving the preorder Γ .

Proof: To avoid triviality, assume $X \neq \emptyset$ and choose an arbitrary point $p \in X$. Define $P = \Sigma p$. As Σ is semiclosed, P is closed. Arbitrarily choose $\pi \in E^1$ and define $f_p: P \rightarrow E^1$ as the constant $f_p = \pi$. Then f_p is trivially bounded, continuous and Γ -preserving. Now choose $\lambda, \lambda' \in E^1$ such that $\lambda < \lambda'$. Defining $A(\lambda)$ and $B(\lambda')$ as in Nachbin's theorem above, the theorem requires only for us to show that $A(\lambda) < B(\lambda')$. Now either (i) $\pi \leq \lambda$ or (ii) $\lambda' \leq \pi$ or (iii) $\lambda < \pi < \lambda'$. In case of (i) or (iii), $B(\lambda') = \emptyset$, so that $I(B(\lambda')) = \emptyset$. In case of (ii), $A(\lambda) = \emptyset$, so that $D(A(\lambda)) = \emptyset$. Thus, in all

cases $D(A(\lambda)) \cap I(B(\lambda')) = \emptyset$, showing that $A(\lambda) < B(\lambda')$. We conclude that there exists a bounded, continuous, real-valued, Γ -preserving $f: X \rightarrow E^1$ with, in fact, $f(P) = \pi$.

3.6 Corollary: (See 4.1.1 below)

4. Applications in the Theory of Preference

This section will illustrate how the methods so far presented may fruitfully be applied in social analysis. The chosen specific area of application is the theory of preference, otherwise known to economists as "utility theory".

4.1 Representation of a Preference Relation:

A celebrated result in this theory is Debreu's [1, Theorem I, p. 162] following first "representation"

Theorem: Let X be a separable connected space completely preordered by a semiclosed relation $\Gamma \subset X^2$. Then there exists a continuous real-valued function $f: X \rightarrow E^1$ preserving Γ (in fact, representing Γ , i.e., obeying $y \in \Gamma x$ iff $f(y) \geq f(x)$).

The first contribution of our results in the previous sections to our present area of application consists in subsuming Debreu's just-stated theorem as a direct corollary of the more general

4.1.1 Corollary: Let X be a connected space completely preordered by a relation $\Gamma \subset X^2$. Then there exists a bounded continuous real-valued representation $f: X \rightarrow E^1$ of Γ iff Γ is semiclosed.

Proof: By 3.4, X is normally preordered by Γ if Γ is semiclosed. If Γ is semiclosed, then so is Σ , whereby 3.5 directly yields a bounded, continuous, Γ -preserving function $f: X \rightarrow E^1$ (which, by completeness of Γ , is, in fact, a representation of Γ), as sought. The converse is obvious.

In comparing 4.1.1 with Debreu's indicated theorem, it will be found that the hypothesis of 4.1.1 is weaker, missing the separability of X , while its consequence appears stronger, guaranteeing a bounded f of the desired sort. Of these differences, it must be remarked, the latter should not be considered important or, for that matter, real, as we can always use the bounded function

$$g = \begin{cases} f/f+1 & \text{if } f \geq 0 \\ f/f-1 & \text{if } f \leq 0 \end{cases}$$

instead of f whenever f happens not to be bounded, and g obviously has all properties desired of f . The absence of the separability assumption for X in 4.1.1, however, must be viewed as a strict improvement with some important "practical" consequences - from a technical viewpoint - for the theoretician concerned with matters of preference. [It is a standard topological fact that a product space $X = \prod_A X_\alpha$ (with the product topology of $\{X_\alpha \mid \alpha \in A\}$) is connected iff each X_α is connected but that the following restrict the product invariance of separability and 2° countability: (1) X is separable iff each

X_α is separable and all but at most 2^{\aleph_0} of the X_α 's consist of single points, where \aleph_0 is the cardinality of the set of natural numbers ("aleph naught"); (2) X is 2° countable iff each X_α is 2° countable and all but \aleph_0 of the X_α 's are indiscrete. (See Marczewski [8] concerning (1).) Thus, for instance, a topological vector space obtained as the product of more than 2^{\aleph_0} copies of the real line is neither separable, nor, indeed, 2° countable, while every convex subset is (in fact arcwise-) connected, so that, in this case Debreu's second representation theorem [1 , Theorem 2, p. 163] is just as inapplicable as his first, while 4.1.1 can be used. Furthermore, interpreting Herstein and Milnor's [7] mixture set in the natural sense of convex set here, there are connected sets in this vector space, notably (from the viewpoint of generalized Kuhn-Tucker theory) the star-shaped sets, which will not be mixture sets, so that an instance is found where the representation theory of [7] - which, incidentally, deals with the case where Γ is a total order - will not apply while 4.1.1 will.]

Applications of 4.1.1 extend also into the next subsection. Another simple fact to be used there but properly belonging under the present heading is the following

4.1.2 Lemma: Let $\Gamma \subset X^2$ be a complete transitive relation on a connected space X , and let $f: X \rightarrow E^1$ be a Γ -preserving real-valued function. f is (a) continuous (representation of Γ) iff Γ is semiclosed and $f(X)$ is connected.

Proof: It is obvious (and stated merely for completeness) that Γ is semiclosed and $f(X)$ is connected if f is continuous. We prove only the converse. For this it suffices to show that the inverse image $f^{-1}(W) \subset X$ of every subbasic open set $W \subset E^1$ is open if Γ is semiclosed and $f(X)$ connected, and we take $W = \{w \in E^1 \mid w > w^*\}$ for some arbitrary $w^* \in E^1$, remarking that the argument will be entirely similar for $W' = \{w \in E^1 \mid w < w^*\}$. If $w^* \notin f(X)$, then connectivity of $f(X)$ implies that either (i) $w^* < w$ for all $w \in f(X)$ or (ii) $w^* > w$ for all $w \in f(X)$. If (i), then $f^{-1}(W) = X$; and if (ii), then $f^{-1}(W) = \emptyset$; in either case $f^{-1}(W)$ is open. Now consider $w^* \in f(X)$, and let $w^* = f(x^*)$. Then $f^{-1}(W) = x^* \Gamma^c$, by the fact that Γ is complete and f Γ -preserving. Therefore, if Γ is semiclosed, then $f^{-1}(W)$ is open, and this completes the proof.

4.2 The Structure of Preference Relations and Their Representations:

Otherwise stated, the topic of the present subsection is that of "aggregation" and, in particular, the "separability" - additively or in general - of utility functions. The immediate motivation is to extend the complete characterization by Gorman [6] of the "separability" and, in general, the structure of utility functions.

This characterization was given by Gorman under the assumptions, among others, that the space of prospects was (topologically) separable and arcwise connected. Arcwise connectivity

was effectively shown [10] to be relaxable to connectivity for Gorman's results, being so for his underlying Lemma 1 [6, p. 387]. It appears that the only reason for postulating separability as an assumption was to ensure the existence of continuous utility functions, invoking Debreu's first representation theorem stated in 4.1. As 4.1.1 now outrules any need for this theorem and for the space of prospects to be separable, one quickly intuits that Gorman's separability assumption may also be eliminated. Here we show that his Lemma 1 can be extended so as to apply whether or not the space of prospects is separable (or arcwise connected), so long as it is connected, and encourage the reader to check that this actually yields a corresponding extension of the whole of Gorman's results in [6], so that his assumptions of separability and arcwise connectivity can, in fact, be diminished to connectivity throughout.

I consider this as opportune a moment as any to indulge in the premature expression of a thought, as fuzzy as it is in my mind, that what we, including Professor Gorman, are looking at is a topic of interest in its own right as having to do with the "structure", in general, of relations and of maps preserving them, deserving at least a glance by specialists in functional equations and semigroups - if, indeed, they had not

already seen through the matter. And, continuing to bleed out this thought, Professor Gorman's analysis of the structure of A (see 4.2.1) is probably a key to more doors than meet the eye in this dimly illuminated hallway.

We now turn to more concrete matters. Throughout, $X = \prod_A X_\alpha$ will be the product of a family $\{X_\alpha \mid \alpha \in A\}$ of spaces, and $\Gamma \subset X^2$ will be a relation on X . Given any $B \subset A$, we will denote $X_B = \prod_B X_\alpha$ and $X^B = X_{B^c}$. Projection of X onto a factor X_B will be denoted by π_B , and projection of X^2 onto X_B^2 will be denoted by π_B^2 . Finally, for each $B \subset A$, we define two relation-valued maps γ and γ_B on X^B by

$$\gamma(x^B) = \Gamma \cap [X_B \times \{x^B\}]^2 \quad \text{and} \\ \gamma_B(x^B) = \pi_B^2(\gamma(x^B)),$$

lower case Latin denoting, as from here on, a generic element of the respective capital [e.g., $x \in X$, $x_B \in X_B$, $x^B \in X^B$, etc.], and x_B denoting $\pi_B(x)$, etc.

4.2.1 Definition: We say that A is semidivisible by B or that B is a sector of A , and we write $A|B$, iff γ_B is a constant function (in which case its constant value on X^B clearly coincides with $\pi_B^2(\Gamma)$, which latter we denote by Γ_B). We say that A is divisible by B , that B divides A , or that B is a complemented sector or factor of A , and we write $A||B$, iff $A|B$ and $A|B^c$. Finally, we define

$$A| = \{B \mid A|B\},$$

$$A|| = \{B \mid A|B\}.$$

The following are clear.

4.2.2 Proposition: $\emptyset, A \in A|| \subset A|$.

4.2.3 Proposition: For any $B, C \subset A$ and any $x = (x_B, x^B) = (x_C, x^C) \in X$,

$$\gamma(x^B \cap C) \supset \gamma(x^B) \cap \gamma(x^C), \quad \text{and}$$

$$\gamma(x^B \cup C) \subset \gamma(x^B) \cap \gamma(x^C).$$

In fact, for any subset $B \subset 2^A$, denoting $\pi_B(x) = x_B$ ($x \in X, B \subset A$),

$$\gamma(x^D) \supset \bigcap_{B \in B} \gamma(x^B) \supset \gamma(x^E),$$

where $D = \bigcap_{B \in B} B$ and $E = \bigcup_{B \in B} B$.

4.2.4 Proposition: $A|$ is closed under arbitrary intersection,

i.e., $B \subset A| \Rightarrow \bigcap_{B \in B} B \in A|$. [Thus, $A|$ together with the partial order \subset of containment is a complete lower semilattice $(A|, \subset)$ with $\text{Inf}(A|) = \bigcap A| = \emptyset$ and $\text{Sup}(A|) = A$. Thus, $A|$ together with the binary operation $\cap : A|^2 \rightarrow A|$ of intersection is a commutative band (band: semigroup of idempotents) $(A|, \cap)$ with identity element A and zero \emptyset .]

Proof: Merely observe that the containment $\gamma(x^D) \supset \bigcap_{B \in B} \gamma(x^B)$ of the last proposition becomes an equality whenever $B \subset A|$. In that case γ_D is a constant function, as γ_B is so for

each $B \in \mathcal{B}$; thus, $D \in \mathcal{A}$. [The parenthetical note is now obvious.]

Thus, if we know that \mathcal{B} is a set of sectors of \mathcal{A} , then we know that the intersection of the members of any subset $C \subset \mathcal{B}$ is also a sector of \mathcal{A} . For $\mathcal{A}||$ we have the obvious

4.2.5 Proposition: $\mathcal{A}||$ is closed under complementation, so that, if B and C are factors of \mathcal{A} , then $B \setminus C$ and $C \setminus B$, as well as B^c , C^c , $B \cap C$ and $B^c \cap C^c = (B \cup C)^c$ are sectors of \mathcal{A} .

We now focus our attention on the case where Γ is a pre-order. In this connection, the following two propositions collect some elementary facts, the proofs of which are straightforward.

4.2.6 Proposition: If a relation $\Gamma \subset X^2$ on a product $X = \prod_A X_\alpha$ satisfies any one of the properties transitivity/reflexivity/symmetry/antisymmetry/completeness, then so does each projection $\Gamma_B = \pi_B^2(\Gamma)$ ($B \subset A$).

4.2.7 Proposition: Let $\Gamma \subset X^2$ be a relation on a product space $X = \prod_A X_\alpha$, such that $\Gamma = \Gamma_B \times \Gamma_C$. Then Γ is transitive/reflexive/symmetric/antisymmetric, respectively, if Γ_B and Γ_C are so. Furthermore, Γ is irreflexive if at least one of the projections Γ_B and Γ_C is so.

Thus, in particular, if Γ is a (complete) preorder, then so is each projection Γ_B ($B \subset A$). The relation between Γ

and its projections Γ_B is stronger for sectors B of A . In fact, one may see that the structure of $A|$ and of $A||$ bear a systematic kinship with the "structure" of Γ , by the latter of which we refer, broadly, to the set of relationships obtaining between Γ and certain of its projections Γ_B . Both the "structure" of Γ and that of the collections $A|$ and $A||$ are related to the "structure" of functions $u: X \rightarrow E^1$ preserving Γ , more particularly to the forms in which such functions can be written in terms of certain functions $u_B: X_B \rightarrow R$. Information gained about either of these "structures" seems to help illuminate the others. For this reason we now set up some apparatus to deal with the "structure" of Γ -preserving functions $u: X \rightarrow E^1$. Let u be such a function for a preorder $\Gamma \subset X^2$, and let $\bar{x} \in X$. We use \bar{x} as a "reference point" to define $\bar{u}_B: X_B \rightarrow R$ by $\bar{u}_B(x_B) = u(x_B, \bar{x}^B)$ ($B \subset A$). It is clear that \bar{u}_B preserves the preorder $\Gamma \cap (X_B \times \{\bar{x}^B\})^2$ on $X_B \times \{\bar{x}^B\}$. In the case where $B \in A|$, and only in this case, however, varying \bar{x}^B in X^B does not alter (the projection onto X_B^2 of) this preorder considered as a relation on X_B . Thus, in this case, and only in this case, we abbreviate \bar{u}_B to u_B .

Some clues involving the form of u and pairwise disjoint families of sectors of A are furnished by the following

4.2.8 Lemma: Let $\Gamma \subset X \times X$ be a complete preorder on a nonempty product $X = \prod_A X_\alpha$, let $u: X \rightarrow R$ be a real-valued representation of Γ , and let $\{B_\eta \mid \eta \in N\}$ be a partition of $A \setminus C$

for some $C \subset A$. Then the following are equivalent.

4.2.8.1 There exists a family $\{v_\eta: X_{B_\eta} \rightarrow \mathbb{R} \mid \eta \in N\}$ of real-valued functions and a function $f: X_C \times \prod_N v_\eta(X_{B_\eta}) \rightarrow \mathbb{R}$ increasing in each v_η ($\eta \in N$), such that u can be expressed as

$$u(x) = f(x_C, \{v_\eta(x_{B_\eta})\}_{\eta \in N}).$$

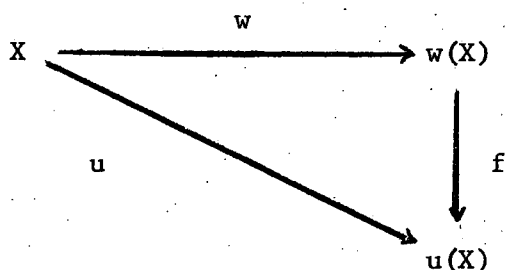
4.2.8.2 For each $\eta \in N$, B_η is a sector of A .

Proof: (ad 4.2.8.1 \Rightarrow 4.2.8.2): Assume 4.2.8.1, and fix attention to an arbitrary $\eta \in N$ and an arbitrary ("reference point") $\bar{x} \in X$. Suppose that $\tilde{x} \in \Gamma \tilde{y}$, i.e., $u(\tilde{x}) \geq u(\tilde{y})$, for some $\tilde{x} = (x_{B_\eta}, \bar{x}_{B_\eta}^B)$, $\tilde{y} = (y_{B_\eta}, \bar{y}_{B_\eta}^B) \in X$. As f is increasing in v_η , we then have $v_\eta(x_{B_\eta}) \geq v_\eta(y_{B_\eta})$. Suppose $x = (x_{B_\eta}, x_{B_\eta}^B)$, $y = (y_{B_\eta}, y_{B_\eta}^B) \in X$. Clearly, the proof rests on being able to show that $u(x) \geq u(y)$. Now $(x_C, \{v_\mu(x_{B_\mu})\}_{\mu \in N})$ can differ from $(y_C, \{v_\mu(y_{B_\mu})\}_{\mu \in N})$ only in so far as $v_\eta(x_{B_\eta})$ differs from $v_\eta(y_{B_\eta})$, since the B_μ 's are pairwise disjoint. But $v_\eta(x_{B_\eta}) \geq v_\eta(y_{B_\eta})$ has already been established. As f is increasing in v_η , this implies that $u(x) \geq u(y)$, i.e., that $x \in \Gamma y$. Thus, γ_{B_η} is identically Γ_{B_η} , whereby $A|B_\eta$. As $\eta \in N$ was arbitrary, we have shown 4.2.8.1 \Rightarrow 4.2.8.2.

(ad 4.2.8.2 \Rightarrow 4.2.8.1): Assume 4.2.8.2, denote $v(x^C) = \{u_{B_\eta}(x_{B_\eta})\}_{\eta \in N}$, and define $w: X \rightarrow X_C \times v(X^C)$ by

26-(a)

$w(x) = (x_C, v(x^C))$. All we need to show is that the diagram



commutes for some function f , and for that it suffices to show that u is constant on the inverse image $w^{-1}(\omega)$ of each $\omega \in w(X)$, since sending $f: \omega \mapsto u(w^{-1}(\omega))$ then defines f as the desired function. [If N were a finite set equal to, say, $M = \{0, \dots, m\}$, then the constancy of u on $w^{-1}(\omega)$ for each $\omega \in w(X)$ would be clear as a consequence of having $A|_{B_\eta}$ for each $\eta \in N$.] To show this in general, i.e., for arbitrary N , we use transfinite induction.

Thus, consider some well-ordering of N , and denote initial segments of elements $\eta \in N$ by $N(\eta)$. For each $\mu \in N$, define

$$B_\mu^* = \bigcup_{N(\mu)} B_\eta \quad \text{and} \quad C_\mu^* = A \setminus B_\mu^*,$$

and the functions w_μ and \bar{w}_μ on X by

$$w_\mu(x) = (x_{C_\mu^*}, \{u_{B_\eta}(x_{B_\eta})\}_{\eta \in N(\mu)}) \quad \text{and}$$

$$\bar{w}_\mu(x) = (x_{C_\mu}, u_{B_\mu}(x_{B_\mu}), \{u_{B_\eta}(x_{B_\eta})\}_{\eta \in N(\mu)}),$$

where $C_\mu = C_\mu^* \setminus B_\mu$. Now we suppose that u is constant on

C

$w_\mu^{-1}(\omega^*)$ for each ω^* of the form $\omega^* = (x_{C_\mu}^*, \{u_{B_\eta}(x_{B_\eta})\}_{\eta \in N(\mu)})$ and show that, given an arbitrary $\omega' = (x_{C_\mu}, \rho_\mu, \{\rho_\eta\}_{\eta \in N(\mu)})$, where $\rho_\lambda \in u_{B_\lambda}(X_{B_\lambda})$ ($\lambda \in N$), u is constant on $w_\mu^{-1}(\omega')$. For given such an ω' , there exists an ω^* whose projection into X_{C_μ} coincides with that of ω' and whose projection into $u_{B_\lambda}(X_{B_\lambda})$, coincides, for each $\lambda \in N(\mu)$, with that of ω' ; furthermore, we may select ω^* so that its projection into X_{B_μ} is a point x_{B_μ} such that $u_{B_\mu}(x_{B_\mu}) = \rho_\mu$. By hypothesis, $u(w_\mu^{-1}(\omega^*))$ is some (constant) $u^\circ \in R$, and, by the way in which ω^* was selected, there is a point $x' \in w_\mu^{-1}(\omega')$ such that $u(x') = u^\circ$. But, by the fact that $A|_{B_\mu}$, u then takes the (constant) value u° on each point $x \in w_\mu^{-1}(\omega')$. By application of the principle of transfinite induction, for each $\omega \in w(X)$, u is constant on $w^{-1}(\omega)$, and this completes the proof.

To economize on proofs which are either obvious or both straightforward and tedious, some further facts are given in the form of an

4.2.9 Exercise: If u in 4.2.8 can be expressed in the form indicated in 4.2.8.1, then

4.2.9.1 for each $\eta \in N$, v_η is increasing in u_{B_η} , so that it represents Γ_{B_η} ;

4.2.9.2 f is "strictly increasing" in $\rho = \{\rho_\eta\}_{\eta \in N}$, where

$\rho_\eta \in v_\eta(X_{B_\eta})$ for each $\eta \in N$, in the sense that (a) if $\rho_\eta \geq \rho'_\eta \in v_\eta(X_{B_\eta})$ for each $\eta \in N$, then $f(x_C, \rho) \geq f(x_C, \rho')$ for each $x_C \in X_C$, and (b) if the hypothesis of (a) holds with, furthermore, $\rho_\lambda > \rho'_\lambda$ for some $\lambda \in N$, then $f(x_C, \rho) > f(x_C, \rho')$ for each $x_C \in X_C$;

4.2.9.3 u can be expressed as

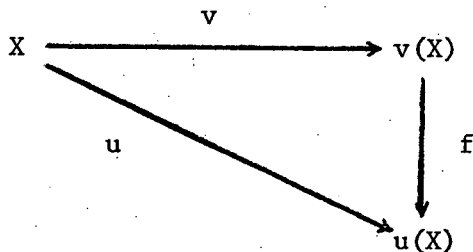
$$u(x) = g(x_C, \{u_{B_\eta}(x_{B_\eta})\}_{\eta \in N})$$

for some g .

So far in this subsection, the discussion invoked no topology. The next lemma, also proved in [10], is concerned with the continuity of the "macroscope" functions f (and g) as in 4.2.8.1 (4.2.9.3). For the purposes of that lemma and some later developments, it is useful to agree on some notation. Accordingly, from now on M will denote the set $\{0, 1, \dots, m\}$ of the first $m + 1$ non-negative integers, and, for each $i \in M$, M^i will denote $M \setminus \{i\}$. Given a family $\{X_i \mid i \in M\}$ of sets indexed by M , we will denote $X = \prod_M X_i$, $X^i = \prod_{M^i} X_j$ for products, and $x_i \in X_i$, $x^i \in X^i$, $x \in X$ for generic elements, while for projections we will adhere to $x = (x_i, x^i)$.

4.2.10 Lemma: Let $X = \prod_M X_i$ be a connected space completely pre-ordered by a semiclosed relation $\Gamma \subset X \times X$, such that $\{\{i\} \mid i \in M^0\} \subset M$. Let $u: X \rightarrow E^1$ preserve Γ , and, for each $i \in M^0$, let $v_i: X_i \rightarrow E^1$ preserve Γ_i , denoting the identity map of X_0 by v_0 . Define $v: X \rightarrow X_0 \times E^m$

by $v(x) = \{v_i(x_i)\}_{i \in M}$ and let f be a function for which the diagram



commutes. If $u(X)$ is connected, then u and f are continuous.

Proof: Assume $u(X)$ connected. Then u is continuous by 4.1.2. Using the apparatus of the proof of 4.1.2, to show that f is continuous it suffices to show that $f^{-1}(W)$ is open for each subbasic open $W \subset E^1$. We take $W = \{w \in E^1 \mid w > w^*\}$ for some $w^* \in E^1$ and show that $f^{-1}(W)$ is open, remarking that the argument is similar for $W' = \{w \in E^1 \mid w < w^*\}$. If $w^* \notin u$, then connectivity of $u(X)$ implies that either (i) $w^* > u(x)$ for all $x \in X$ or (ii) $w^* < u(x)$ for all $x \in X$. If (i) holds, then $u^{-1}(W) = X$, so that commutativity of the diagram yields $f^{-1}(W) = v(u^{-1}(W)) = v(X)$, whereby $f^{-1}(W)$ is open. In case of (ii), $u^{-1}(W) = \emptyset = v(u^{-1}(W)) = f^{-1}(W)$, so that $f^{-1}(W)$ is again open. So, assume $w^* \in u(X)$ and, say, $w^* = u(x^*)$. As u preserves Γ and Γ is complete, we have $u^{-1}(W) = x^* \Gamma^c$, which is open by semiclosedness of Γ (or by continuity of u) and is connected by connectivity of

X. Hence, for each $i \in M$, the projection

$P_i = \pi_{X_i}(u^{-1}(W)) \subset X_i$ is open and connected. Now $v_o(P_o)$

is open trivially. On the other hand, for $i \in M^o$, as

v_i preserves Γ_i , we have $v_i(P_i) = \{w \in v_i(X_i) \mid w >$

$v_i(x_i^*)\}$, which is open. Thus, $v(u^{-1}(W)) = \prod_M v_i(P_i)$ is

open, so that the commutation $f^{-1} = v \circ u^{-1}$ yields

$f^{-1}(W)$ open, from which we conclude f to be continuous.

This completes the proof.

4.2.11 Corollary: Let $u: X \rightarrow E^1$ be a continuous representation of a complete semiclosed relation $\Gamma \subset X \times X$ on a connected product space $X = \prod_M X_i$. Then the following are equivalent.

4.2.11.1 for each $i \in M^o$, $\{i\} \in M$;

4.2.11.2 for some family $\{v_i: X_i \rightarrow E^1 \mid i \in M^o\}$ and some continuous $f: X_o \times \prod_{M^o} v_i(X_i) \rightarrow E^1$ increasing in each v_i ($i \in M^o$), u can be expressed as

$$u(x) = f(x_o, v_1(x_1), \dots, v_m(x_m)).$$

Furthermore, in this case, f is "strictly increasing"

(see 4.2.9.2) in $\rho = \{\rho_i\}_{i \in M^o}$, where $\rho_i \in v_i(X_i)$

for each $i \in M^o$, and, for each $i \in M^o$, v_i is a representation of Γ_i .

Proof: The stated equivalence directly follows from the conjunction of 4.2.8 and 4.2.10. The rest directly follows from 4.2.9.1-2.

This last corollary extends the fundamental Lemma 1 on which

Gorman erects his formidably complete characterization of "The Structure of Utility Functions" [6] in the case where X is arcwise connected and separable. [It also extends the result which Debreu set out to prove in [2, lines 5-18, p. 22].] Our extension here consists of relaxing the assumptions on X to connectivity alone. It was shown earlier [10] that arcwise connectivity could be relaxed to connectivity and promised there that the separability assumption, too, could be deleted, as its role consisted of allowing application of Debreu's first representation theorem, stated above as a corollary to 4.1.1. Given 4.1.1, we now have no need for the separability assumption on X , and the earlier promise is met.

The main question, however, is whether the assumptions of arcwise connectivity and separability, made throughout by Gorman [6] can be relaxed throughout [6] to connectivity alone. Upon studying the mentioned paper, I find it safe to state that my conjecture is Yes. If the proofs of Gorman had to be modified extensively to support this conjecture, then here would be a good place to do that. Fortunately, this does not appear to be the case, so that the (well-advised) reader who also reads Gorman's paper will find, I think, that Gorman's characterization applies so long as X is connected. [In fact, if Gorman's Lemma 1 is replaced by 4.2.11, the only necessary modification I can find in his proofs is in that of his basic theorem (Theorem 1), where 'arc connected' in the first line of 2.18 (p. 372) should be replaced by 'connected', in which

case the theorem, and hence the paper as a whole, will be seen to be preserved.] Thus, the best service to be offered here to the reader seems to be to recommend Gorman's mentioned paper. Hence, we let this last sentence be a pointer toward the cited study by Gorman which has been the source of motivation for this subsection and the validity of which has now been extended.

4.3 Social Optimum and Consensus - A Simple Existence Result:

The result (4.3.2) we aim to demonstrate here is, indeed, very simple. We are given a family $\{\Gamma_\alpha \subset X^2 \mid \alpha \in A\}$ of transitive relations Γ_α , each on the same set X . For each subset $B \subset A$, Γ_B now denotes the intersection $\Gamma_B = \bigcap_B \Gamma_\alpha$, and we simplify $\Gamma_{\{\alpha\}}$ to Γ_α and Γ_A to Γ . With respect to Γ , the set of maximal points (supremal points) in X is denoted by $\text{Max } X$ ($\text{Sup } X$), where the standard definitions

$$\text{Max } X = \{x \in X \mid (x, y) \in \Gamma \Rightarrow (y, x) \in \Gamma\},$$

$$\text{Sup } X = \{x \in X \mid X \times \{x\} \subset \Gamma\}$$

apply. Obviously,

4.3.1 Proposition: $\text{Max } X \supset \text{Sup } X$.

The economist will recognize that $\text{Max } X$ is the set of Pareto-optimal points. $\text{Sup } X$, on the other hand, is a rather more interesting set, consisting of all those points which are "superior" to every point in X unanimously from the viewpoint of each Γ_α ($\alpha \in A$). Of course, $\text{Sup } X$ may very well be

empty, even if $\text{Max } X$ is not. In 4.3.2 we show conditions under which both are non-empty.

Toward that, we agree to say that a relation Γ_α on a topological space X is upper semiclosed iff $\Gamma_\alpha x$ is closed for each $x \in X$. By the condition of Consensus in the Small (CS), we mean

CS: For each pair (B, Y) of nonempty finite subsets $B \subset A$ and $Y \subset X$,

$$\bigcap_{y \in Y} \Gamma_B y \neq \emptyset.$$

Finally, we are able to state and prove the intended

4.3.2 Proposition: If X is compact and each (transitive relation) $\Gamma_\alpha \subset X^2$ is upper semiclosed ($\alpha \in A$) while the condition CS is satisfied, then $\text{Sup } X \neq \emptyset$.

Proof: As each Γ_α is transitive, so is Γ . Hence,

$\text{Sup } X = \bigcap_{X \times A} \Gamma_\alpha x$. Clearly, CS implies the finite intersection property that $\bigcap_F \Gamma_\alpha x \neq \emptyset$ for each finite $F \subset X \times A$. If $\Gamma_\alpha x$ is closed for each $(x, \alpha) \in X \times A$, then CS combined with compactness of X implies that $\bigcap_{X \times A} \Gamma_\alpha x \neq \emptyset$, as to be shown.

We may refer to $\text{Sup } X \neq \emptyset$ as the condition of Consensus in the Large (CL). In that case, 4.3.2 reads to yield a sufficient condition, namely that X is compact and each Γ_α upper semiclosed ($\alpha \in A$), under which $\text{CS} \Rightarrow \text{CL}$, i.e.,

$CS \Leftrightarrow CL$, as $CL \Rightarrow CS$ is always true. In practice, X is often compact while the Γ_α 's are accustomed to be assumed semiclosed complete preorders, hence certainly upper semiclosed transitive relations. Perhaps this yields a practical relevancy to 4.3.2. I do not know whether it is worthwhile obtaining corresponding results by considering the case with a measure space of agents, following Debreu [3] and others.

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"Order, Topology and Preference" Errata

1. Page 9, line 12 should read:

"decreasing, since $U \cap V = \emptyset$. Thus, $(U \times V) \cap \Gamma = \emptyset$, whereby

2. Page 11, 3.1 should read:

Lemma: Let X be a connected space and $\Gamma^c \subset X^2$ a semiclosed complete antisymmetric relation. Then Γ is transitive, hence Γ^c is a total order.

3. Page 12, 3.2 should read:

3.2 Theorem: A complete antisymmetric relation on a connected space is a closed total order iff it is semiclosed.

Proof: "Only if" follows directly from 2.1.3. To prove "if", let X be a connected space and $\Gamma \subset X^2$ a semiclosed, complete, transitive relation on X . By 3.1, Γ is a total order. We now show that Γ^c is open. This is trivially so if $\Gamma^c = \emptyset$, so suppose $(y, x) \in \Gamma^c$, i.e., $x \in \Gamma^c y$ and $y \in x\Gamma^c$. As Γ is semiclosed, $x\Gamma^c \cup \Gamma^c y$ is a union of two nonempty open sets. Furthermore, given an arbitrary $z \in X$, if $z \notin x\Gamma^c$, then $z \in x\Gamma$ and, by transitivity of Γ , $z \in \Gamma^c y$. Thus, $X \subset x\Gamma^c \cup \Gamma^c y$. X being connected, there must, therefore, be some $z \in x\Gamma^c \cap \Gamma^c y$. Now $(z\Gamma^c) \times (\Gamma^c z)$ is a nbd of (y, x) . To see that this nbd is contained in Γ^c , let $a \in z\Gamma^c$ and $b \in \Gamma^c z$. By completeness of Γ , we then have $a \in \Gamma z$ and $b \in z\Gamma$ (i.e., $z \in \Gamma b$), so that transitivity of Γ implies $a \in \Gamma b$. Thus, $b \in \Gamma a$ would imply, by antisymmetry of Γ , that $b = a$, contradicting the obvious fact that $z\Gamma^c \cap \Gamma^c z = \emptyset$. Hence, $b \in \Gamma^c a$, i.e., $(a, b) \in \Gamma^c$. Thus, $(z\Gamma^c) \times (\Gamma^c z) \subset \Gamma^c$, showing that Γ^c is open, i.e., that Γ is closed.

4. Page 13, lines 11 and 12 should read:

Proof: That Γ is a closed total order follows directly from 3.2. Then 2.4 implies that X is Hausdorff.

5. Page 15, line 9 should read:

reflexivity of Γ), connectivity of X implies that

6. Page 15, lines 18 and 19 should read:

$y\Gamma^c$ an open increasing nbd of B , and $\Gamma^c y \cap y\Gamma^c = \emptyset$ by completeness of Γ , Hence, $A \ll B$, showing that X is normally

7. Page 18, line 14:

there should be a minus sign (-) before $f/f-1$ if $f \leq 0$